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Second Variation and Generalized Jacobi Equations for Curvature Invariants

0. INTRODUCTION

For a number of physical motivations which have no direct interest here, since the early days of Relativity attention has been paid to the so-called «non linear theories of gravitation». In contrast to standard «General Relativity», which is governed by the Einstein-Hilbert Lagrangian, i.e., by the scalar curvature R of a Lorentzian metric g , these are governed instead, through a variational principle, by Lagrangians which are either nonlinear functions of R or even depend on higher curvature invariants of g . In particular, renewed attention has grown in the recent past about theories governed by general Lagrangian densities of the form $L = f(R) \sqrt{g}$ (see [1] for a comprehensive review, as well as references quoted therein), or by Lagrangians quadratic in the curvature, i.e. depending linearly on the quadratic invariants $\|Ric(g)\|^2$ and $\|Riem(g)\|^2$, $Ric(g)$ and $Riem(g)$ being respectively the Ricci tensor and the Riemann tensor of the metric (see, e.g., [2], [3], [4] and references quoted therein).

A particularly relevant question for physical applications (which range from inflationary cosmology to quantization procedures and involve singularity issues, as well as the problems of linearized theories, weak-energy limit in string theory and the very notions of gravitational energy for nonlinear gravity) is the stability of critical points, i.e., of solutions of field equations. As is well known (see also [5] and [6] for a revisitation of this issue in the case of first order Lagrangian theories) these depend on the behaviour of the se-

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cond variation of the action functional (near critical points) and of those special deformations which are governed by generalized Jacobi equations along critical sections.

Motivated by these and in view of further applications, in this paper we shall thoroughly address, in a self-contained exposition, all the relevant formulae for the first and second variation of quadratic curvature Lagrangians in a n -dimensional manifold M with a metric g (of any signature) and a linear connection Γ (which, for simplicity will be assumed to be symmetric). In order to let the results to be applicable both to purely metric and to metric-affine theories (i.e., to apply the improperly called «Palatini method») we shall consider both the case in which g and Γ are independent and the case in which Γ is a priori assumed to be the Levi-Civita connection of g . The case of more complicated Lagrangians such as $f(R)$, $f(\|Ric\|^2)$ and $f(\|Riem\|^2)$ will be considered in a further paper [7]. For simplicity in this paper we shall consider the first – and second-variation, together with the appropriate Jacobi equations, for the Lagrangian functions rather than the corresponding densities. It is in fact known that all variational relations involving a Lagrangian $f(j^2g)$ have a counterpart in terms of the corresponding density $f(j^2g)\sqrt{8}$, but calculations are shorter (see [7] Section 1.2 and [8]).

1. SOME USEFUL OPERATORS

Let M be a C^∞ -differentiable n -dimensional manifold and (x^u) be a local coordinate system. Let $I'_s(M)$ be the module of tensorfields of type (r,s) ; we have $\mathbf{X}(M) = I'_0(M)$ and $\mathfrak{X}(M) = I'_0(M)$, $\mathfrak{N}(M)$ and $\mathfrak{F}(M)$ being the Lie algebra of vectorfields and the ring of differentiable functions on M , respectively. It is well known that some linear operators play an important role in the Calculus of Variations, when Lagrangians involving the curvature tensorfields are used (see e.g., [9] and [10]). In this Section we shall list some of these operators and determine some of their properties related to the «duality» with respect to contractions and to the commutativity with respect to a suitable derivation. All operators and relations appearing in this paper are globally defined; however, for the sake of simplicity and especially in view of physical applications, we shall mostly use a local notation and give equations in their component form.

The first operator we need will be denoted by $\mathcal{A}: \mathfrak{F}_2^1(M) \rightarrow \mathfrak{F}_2^1(M)$. It is defined in any local chart by putting

$$\mathcal{A}(t)_{\beta\gamma}^\alpha = t_{\beta\gamma}^\alpha - \delta_{(\beta}^\alpha t_{\gamma)\sigma}^\sigma, \quad (1.1)$$

for each $t \in \mathfrak{S}_2^1(M)$ having local components $t_{\beta\gamma}^\alpha$. Here and in the sequel round brackets grouping indices denote symmetrization and δ_β^α are the Kronecker symbols. For later convenience we shall often make use of the following abuse of notation. If $f: \mathfrak{S}_q^p(M) \rightarrow \mathfrak{S}_q^p(M)$ is a linear operator on tensors we write in components:

$$f\left(t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_q}\right) = (f(t))_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_q}$$

for example, with this notation equation (1.1) is written in the form:

$$\mathcal{A}(t)_{\beta\gamma}^\alpha \equiv \mathcal{A}(t_{\beta\gamma}^\alpha) = t_{\beta\gamma}^\alpha - \delta_{(\beta}^\alpha t_{\gamma)\sigma}^\sigma \quad (1.1')$$

The operator \mathcal{A} can be extended to an operator on the components of $\Gamma_{\beta\gamma}^\alpha$ of a linear connection Γ on M and one puts (cfr. [10]):

$$\mathcal{A}(\Gamma)_{\beta\gamma}^\alpha \equiv u_{\beta\gamma}^\alpha(\Gamma) = u_{\beta\gamma}^\alpha \equiv \Gamma_{\beta\gamma}^\alpha - \delta_{(\beta}^\alpha \Gamma_{\gamma)\sigma}^\sigma \quad (1.2)$$

The family $(u_{\beta\gamma}^\alpha)$ determines a new coordinate system in the bundle $\mathfrak{S}(M)$ of linear connections, being \mathcal{A} a bijection. In fact, the inverse of \mathcal{A} , denoted by \mathcal{A}^{-1} exists for each $n > 1$ and it is defined by:

$$\mathcal{A}(t)_{\beta\gamma}^\alpha = t_{\beta\gamma}^\alpha + \frac{2}{1-n} \delta_{(\beta}^\alpha t_{\gamma)\sigma}^\sigma \quad (1.3)$$

Let $t \in \mathfrak{S}_s^r(M)$ be any family of tensorfields depending differentiably on $\varepsilon \in]-a, a[$, with $a > 0$ any real number. We define the first variation dt by:

$$(\delta t)_{\beta_1 \dots \beta_r}^{\alpha_1 \dots \alpha_r} = \frac{(\delta t)_{\beta_1 \dots \beta_r}^{\alpha_1 \dots \alpha_r}}{\partial \varepsilon} \Big|_{\varepsilon=0} \quad (1.4)$$

From linearity it follows immediately:

$$\delta \mathcal{A}(t)_{\beta\gamma}^\alpha = \mathcal{A}(\delta t_{\beta\gamma}^\alpha) \quad (1.5)$$

Finally, the *adjoint* (or *dual*) of A is the linear operator $\mathcal{A}^*: \mathfrak{S}_1^2(M) \rightarrow \mathfrak{S}_1^2(M)$ defined by:

$$\langle t, \mathcal{A}^*(s) \rangle = \langle A(t) \mid s \rangle, \quad \forall t \in \mathfrak{S}_2^1(M) \quad \forall s \in \mathfrak{S}_1^2(M) \quad (1.6)$$

where $\langle, \rangle: \mathfrak{S}_q^p(M) \times \mathfrak{S}_p^q(M) \rightarrow \mathfrak{F}(M)$ is the standard duality for any pair (p, q) of integers. Locally, \mathcal{A}^* operates in the following way:

$$\mathcal{A}^*(t)_{\alpha}^{\beta\gamma} \equiv \mathcal{A}^*(t)_{\alpha}^{\beta\gamma} = t_{\alpha}^{\beta\gamma} - \delta_{\alpha}^{\gamma} t_{\sigma}^{(\sigma\beta)} \quad (1.7)$$

\mathcal{A}^* is also a bijection and its inverse operator coincides with the adjoint operator of \mathcal{A} for $n > 1$. This operator will be denoted by $\tilde{\mathcal{A}}$ and it is locally defined by:

$$\mathcal{A}^*(t)_{\alpha}^{\beta\gamma} = t_{\alpha}^{\beta\gamma} + \frac{2}{1-n} \delta_{\alpha}^{\gamma} t_{\sigma}^{(\sigma\beta)} \quad (1.8)$$

The three operators above commute with δ i.e.:

$$\delta\mathcal{A} = \mathcal{A}\delta \quad \delta\mathcal{A}^* = \mathcal{A}^*\delta \quad \delta\tilde{\mathcal{A}} = \tilde{\mathcal{A}}\delta \quad (1.9)$$

A further operator we need is $\overline{\mathcal{A}}: \mathfrak{S}_2^1(M) \rightarrow \mathfrak{S}_2^1(M)$, defined by:

$$\overline{\mathcal{A}}(t)_{(\beta\gamma)}^{\alpha} \equiv \overline{\mathcal{A}}(t)_{\beta\gamma}^{\alpha} = 2 t_{\beta\gamma}^{\alpha} - \delta_{\beta}^{\alpha} t_{\gamma\sigma}^{\sigma} \quad (1.10)$$

The operator $\overline{\mathcal{A}}$ satisfies the same properties of \mathcal{A} and its symmetric part is such that:

$$\overline{\mathcal{A}}(t)_{(\beta\gamma)}^{\alpha} = \mathcal{A}(t)_{\beta\gamma}^{\alpha} + t_{(\beta\gamma)}^{\alpha} \quad (1.11)$$

The adjoint operator $\overline{\mathcal{A}}^*: \mathfrak{S}_1^2(M) \rightarrow \mathfrak{S}_1^2(M)$ is given by:

$$\overline{\mathcal{A}}^*(t)_{\alpha}^{\beta\gamma} \equiv \overline{\mathcal{A}}^*(t)_{\alpha}^{\beta\gamma} = 2 t_{\alpha}^{\beta\gamma} - \delta_{\alpha}^{\gamma} t_{\sigma}^{\sigma\beta} \quad (1.12)$$

Let now $\mathfrak{B}: \mathfrak{S}_3^0(M) \rightarrow \mathfrak{S}_3^0(M)$ be the linear operator defined by:

$$2\mathfrak{B}(t_{\alpha\beta\gamma}) = t_{\alpha\beta\gamma} - t_{\beta\gamma\alpha} + t_{\gamma\alpha\beta} \quad (1.13)$$

for any $t \in \mathfrak{S}_3^0(M)$ and $t = t_{\alpha\beta\gamma} dx^{\alpha} \otimes dx^{\beta} \otimes dx^{\gamma}$.

A similar operator can be defined on contravariant tensors in $I_3^0(M)$ and will be denoted by the same letter \mathfrak{B} . The adjoint operator $\mathfrak{B}^*: I_3^0(M) \rightarrow I_3^0(M)$ is defined by:

$$2\mathfrak{B}^*(t^{\alpha\beta\gamma}) = t^{\alpha\beta\gamma} + t^{\beta\gamma\alpha} + t^{\gamma\alpha\beta} \quad (1.14)$$

for any $t \in \mathfrak{S}_3^0(M)$. Obviously, adjointness is considered with respect to the standard duality. Finally, both \mathfrak{B} and \mathfrak{B}^* commute with δ .

Let now $k \in \mathfrak{S}_0^2(M)$ be a symmetric tensorfield, having k^{ab} as local components. For each $t \in \mathfrak{S}_2^1(M)$, with local components $t_{\gamma}^{\alpha\beta}$, we have:

$$\begin{aligned} \text{i) } k^{\alpha\beta} \mathcal{A}(t_{\alpha\beta}^{\gamma}) &= k^{\alpha\beta} t_{\alpha\beta}^{\gamma} - k^{\gamma\beta} t_{\beta\sigma}^{\alpha} \\ \text{ii) } \mathcal{A}(t_{\mu\nu}^{\lambda}) k^{\rho\nu} \overline{\mathcal{A}}(t_{\rho\lambda}^{\mu}) &= 2t_{\rho\lambda}^{\mu} k^{\rho\nu} t_{\rho\lambda}^{\mu} - 2t_{\lambda\sigma}^{\sigma} t_{\rho\nu}^{\lambda} k^{\rho\nu} \end{aligned} \quad (1.15)$$

If t is symmetric with respect to the covariant indices, then (1.15 ii) becomes:

$$\mathcal{A}(t_{\mu\nu}^{\lambda}) k^{\rho\nu} \overline{\mathcal{A}}(t_{\rho\lambda}^{\mu}) = 4 t_{\lambda[\nu}^{\sigma} t_{\sigma]\rho}^{\lambda} k^{\rho\nu} \quad (1.15')$$

Let us consider a tensorfield $t \in \mathfrak{S}_3^0(M)$, with local components, $t_{\alpha\beta\gamma}$ symmetric with respect to the last two indices β and γ . Then the following identities hold:

$$\begin{aligned} \text{i) } 2 k^{\mu\nu} \mathfrak{B}(t_{\mu\alpha\nu}) &= 2 k^{\mu\nu} t_{\mu\alpha\nu} - k^{\mu\nu} t_{\alpha\mu\nu} \\ \text{ii) } 2 k^{\alpha\nu} \mathfrak{B}(t_{\mu\alpha\nu}) &= k^{\alpha\nu} t_{\mu\alpha\nu} \\ \text{iii) } 2 k^{\mu\alpha} \mathfrak{B}(t_{\mu\alpha\nu}) &= k^{\mu\alpha} t_{\nu\mu\alpha} \\ \text{iv) } 2\mathfrak{B}(t_{\mu\alpha\nu}) k^{\alpha\lambda} k^{\mu\beta} k^{\nu\rho} \mathfrak{B}(t_{\rho\beta\lambda}) &= t_{\mu\alpha\nu} k^{\mu\beta} k^{\alpha\lambda} k^{\nu\rho} B(t_{\rho\beta\lambda}) \end{aligned} \quad (1.16)$$

We also need a bilinear operator $F: \mathfrak{S}_0^2(M) \times \mathfrak{S}_2^1(M) \rightarrow \mathfrak{S}_2^1(M)$, defined by:

$$2 F(k, t)_{\beta\gamma}^{\alpha} = k^{\alpha\rho} \mathfrak{B}(t_{\beta\rho\gamma}) \quad (1.17)$$

If Γ is an arbitrary linear connection on M and ∇_{μ} the corresponding covariant derivative operator we define a new operator $F_{\Gamma}: \mathfrak{S}^2(M) \times \mathfrak{S}_0^2(M) \rightarrow \mathfrak{S}_2^1(M)$ by putting:

$$F_{\Gamma}(k, m) = F(k, \nabla m), k \in \mathfrak{S}_0^2(M), m \in \mathfrak{S}_0^2(M) \quad (1.18)$$

Locally:

$$2 F_{\Gamma}(k, m)_{\beta\gamma}^{\alpha} = k^{\alpha\rho}(\nabla_{\beta} m_{\rho\gamma} - \nabla_{\rho} m_{\gamma\beta} + \nabla_{\gamma} m_{\beta\rho}) \quad (1.18')$$

In particular, if g is a pseudo-Riemannian metric and Γ_g is the Levi-Civita connection of g , we set for simplicity:

$$\mathcal{F}_g(m) = \mathcal{F}_{\Gamma_g}(g^*, m) \quad (1.19)$$

where g^* is the contravariant metric. This defines a new operator $\mathcal{F}_g: \mathfrak{S}_2^0(M) \rightarrow I_2^1(M)$ which will also be useful later. Let us finally remark that the operator B defined by (1.13) can be called the *d-adjoint operator* with respect to the operator (1.18), since the following holds:

$$\mathcal{F}_{\Gamma}(k, m)_{\beta\gamma}^{\alpha} k^{\beta\sigma} k^{\gamma\tau} t_{\alpha\sigma\tau} = \nabla_{\beta}[m_{\rho\gamma} k^{\rho\alpha} k^{\beta\sigma} k^{\gamma\tau} \mathfrak{B}(t_{\alpha\sigma\tau})] - m_{\rho\gamma} \nabla_{\beta}[k^{\rho\alpha} k^{\beta\sigma} k^{\gamma\tau} \mathfrak{B}(t_{\alpha\sigma\tau})] \quad (1.20)$$

for any $k \in \mathfrak{S}_0^2(M)$, $m \in \mathfrak{S}_2^0(M)$, $t \in \mathfrak{S}_3^0(M)$ with k symmetric. Notice in fact that the first term of the left hand side of (1.20) decomposes into the sum of a divergence plus a term which can be recast into a divergence at least if $\nabla k = 0$. This remark will be useful for later applications.

2. THE FIRST VARIATION

In this Section we will recall the first variation of some actions involving curvature tensorfields, most of which are well known and useful for applications. On the manifold M we fix a family $g_{\epsilon} \in \mathfrak{S}_2^0$ of metric tensorfields, whose local components will be denoted by $g_{\mu\nu}$, and a family of connections Γ_{ϵ} , whose local components will be denoted by $\Gamma_{\beta\gamma}^{\alpha}$, assuming that both families depend differentiably on a parameter $\epsilon \in]-a, a[$, $a > 0$. The components of $\Gamma_{\beta\gamma}^{\alpha}$ of Γ_{ϵ} split, with respect to the lower indices, in their symmetric part $\tilde{\Gamma}_{\beta\gamma}^{\alpha}$ and in their skew-symmetric part $\frac{1}{2} \Gamma_{\beta\gamma}^{\alpha}$, where $T_{\beta\gamma}^{\alpha}$ are the local components of the torsion. Hereafter, whenever a quantity S depends on Γ the notation \tilde{S} will be used when referring to the corresponding function of $\tilde{\Gamma}$. We also set:

$$h = \delta g \in \mathfrak{S}_2^0(M) \quad F = \delta \tilde{\Gamma} \in \mathfrak{S}_2^1(M) \quad A = \mathcal{A}(F) \in \mathfrak{S}_2^1(M) \quad (2.1)$$

i.e., in components

$$\begin{aligned} h_{\mu\nu} &= \delta g_{\mu\nu} & F_{\beta\gamma}^\alpha &= \delta \tilde{\Gamma}_{\beta\gamma}^\alpha \\ A_{\beta\gamma}^\alpha &= \mathcal{A}(\delta \tilde{\Gamma}_{\beta\gamma}^\alpha) = \mathcal{A}(F_{\beta\gamma}^\alpha) \end{aligned} \quad (2.1')$$

We set also:

$$h^* = \delta g^* \in \mathfrak{S}_0^2(M) \quad (2.2)$$

We recall that the first variation h^* is related to $h = \delta g$ by the rule:

$$(h^*)^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta} \quad (2.2')$$

For notational simplicity we shall write $h^{\mu\nu}$ in place of $(h^*)^{\mu\nu}$. Notice that, if Γ_ϵ is the Levi-Civita connection of g_ϵ , then as $F = \tilde{\mathcal{F}}_g(h)$ given by (1.19) (cfr. [10]).

Let *Ric* and *Riem* be the Ricci tensor and the curvature tensor of Γ_ϵ (ϵ understood), with local components $R_{\alpha\beta}$ and $R_{\beta\gamma\lambda}^\alpha$. The Ricci tensorfield \hat{Ric} splits into its symmetric part \tilde{Ric} and its skew-symmetric part \hat{Ric} whose local components will be denoted respectively by $\tilde{R}_{\mu\nu}$ and $R_{\mu\nu}$. The scalar curvature $\mathfrak{R} = \mathfrak{R}(g, \Gamma) = g^{\mu\nu} \mathfrak{R}_{\mu\nu}(\Gamma)$ is the total contraction of g^* with *Ric* (again ϵ understood). Finally let us recall some formulae which will be often used in the following:

$$\begin{aligned} \delta(\nabla_\nu X^\lambda) &= \nabla_\nu(\delta X^\lambda) + F_{\alpha\nu}^\lambda X^\alpha - \frac{1}{2}(\delta T_{\alpha\nu}^\lambda) X^\alpha \\ \delta R_{\mu\nu}^\lambda &= \nabla_{[\nu} F_{\sigma]}^\lambda - T_{\nu\sigma}^\rho F_{\rho\mu}^\lambda - \nabla \delta T_{\sigma/\mu}^\lambda + \frac{1}{2} T_{\nu\sigma}^\rho \delta T_{\rho\mu}^\lambda \end{aligned}$$

where X_ϵ is a family of vectorfields having X^α as local components.

We shall first consider the «Palatini formalism», in which no relation between g_ϵ and Γ_ϵ is assumed a priori. We define then the four invariants I_a ($a = 1, \dots, 4$) by setting:

$$i) I_1 = \mathfrak{R}$$

$$\begin{aligned}
\text{ii) } I_2 &= \|\check{Ric}\|^2 = \check{Ric}_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} \check{R}_{\alpha\beta} \\
\text{iii) } I_3 &= \|\hat{Ric}\|^2 = \hat{Ric}_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} \hat{R}_{\alpha\beta} \\
\text{iv) } I_4 &= \|Riem\|^2 = Riem^\alpha_{\mu\beta\nu\sigma} g_{\alpha\gamma} g^{\mu\sigma} g^{\beta\lambda} g^{\nu\rho} R^\gamma_{\sigma\lambda\rho}
\end{aligned} \tag{2.3}$$

The decomposition $Ric = \hat{Ric} + \check{Ric}$ is orthogonal, in the sense that:

$$\|Ric\|^2 = \|\hat{Ric}\|^2 + \|\check{Ric}\|^2$$

being:

$$\|Ric\|^2 = R_{\mu\nu} g^{\mu\alpha} g^{\beta\lambda} R_{\alpha\beta}$$

With this notation, a simple calculation gives:

$$\begin{aligned}
\delta\mathfrak{H} &= \nabla_\lambda X^\lambda_{(1)} + h^{\mu\nu} \check{R} - A^\lambda_{\mu\nu} [\nabla_\lambda g^{\mu\nu} + \tilde{A} (g^{\rho\mu} T^\nu_{\lambda\rho})] + \\
&\quad + \frac{1}{2} \delta T^\lambda_{\mu\rho} [\delta^\rho_\lambda \nabla_\nu g^{\mu\nu} - g^{\mu\nu} T^\lambda_{\mu\rho}]
\end{aligned} \tag{2.4}$$

with

$$X^\lambda_{(1)} = X^\lambda_{(1)}(g, \delta T) = g^{\mu\nu} A^\lambda_{\mu\nu} - \frac{1}{2} g^{\mu\nu} \delta T^\rho_{\mu\rho} \tag{2.5}$$

The restriction of (2.4) to the case of torsion-free connections gives the well known formula:

$$\delta\mathfrak{H} = \nabla_\lambda \check{X}^\lambda_{(1)} + h^{\mu\nu} \check{R}^\lambda_{\mu\nu} - A^\lambda_{\mu\nu} \nabla_\lambda g^{\mu\nu} \tag{2.6}$$

with

$$\check{X}^\lambda_{(1)} = \check{X}^\lambda_{(1)}(g^{\mu\nu}, \delta T) = g^{\mu\nu} A^\lambda_{\mu\nu} \tag{2.7}$$

We will use the metric g and the dual metric g^* to raise and lower tensor indices, replacing as usual the index moved by a dot. Let us remark that, according to

our definition (2.2') above, the components $h^{\mu\nu}$ of h^* could also be denoted $-h^{\mu\nu}$. (our notation is hence simpler, but we have to warn the reader that, accordingly, $h_{\mu\nu}$ is not the lowered form of $h^{\mu\nu}$, from which it differs by a sign). Moreover, for notational simplicity, to avoid the use of dots when working on a single index of a skew-symmetric tensor of rank (0,2) or (2,0) we agree to raise or lower just the first index, writing, e.g., R_v^μ in place of R_v^μ .

An easy calculation then gives:

$$\delta(\|\check{\text{Ric}}\|^2) = \nabla_\lambda X_{(2)}^\lambda + 2h^{\mu\nu} \check{R}_{\mu\sigma} \check{R}_\nu^\sigma - 2A_{\mu\sigma}^\lambda [\nabla_\lambda \check{R}^{\mu\sigma} + \check{A}^*(\check{R}^{\mu\rho} T_{\lambda\rho}^\sigma)] + \quad (2.8)$$

$$(\delta T_{\mu\rho}^\lambda) [\delta_\lambda^\rho \nabla_\sigma \check{R}^{\mu\sigma} - T_{\lambda\sigma}^\rho \check{R}^{\mu\sigma}]$$

with

$$X_{(2)}^\lambda = X_{(2)}^\lambda(g, \delta T) = 2A_{\mu\rho}^\lambda \check{R}^{\mu\sigma} - (\delta T_{\mu\sigma}^\rho) \check{R}^{\mu\lambda} \quad (2.9)$$

The above reduces, when Γ_ϵ is a family of torsion-free connections, to the well known formula:

$$\delta(\|\check{\text{Ric}}\|^2) = \nabla_\lambda \check{X}_{(2)}^\lambda + 2h^{\mu\nu} \check{R}_{\mu\sigma} \check{R}_\nu^\sigma - 2A_{\mu\sigma}^\lambda \nabla_\lambda \check{R}^{\mu\sigma} \quad (2.10)$$

with

$$\check{X}_{(2)}^\lambda = \check{X}_{(2)}^\lambda(g, \delta\Gamma) = 1A_{\mu\sigma}^\lambda \check{R}^{\mu\sigma} \quad (2.11)$$

For the skew-symmetric part of the Ricci tensorfield we have instead:

$$\delta(\|\check{\text{Ric}}\|^2) = \nabla_\lambda X_{(3)}^\lambda - 2h^{\mu\nu} \hat{R}_{\mu\sigma} \hat{R}_\nu^\sigma - 2F_{\sigma\rho}^\lambda \left[\delta_\lambda^\rho \nabla_\mu \hat{R}^{\mu\sigma} - T_{\lambda\mu}^\rho \hat{R}^{\mu\sigma} \right] - \quad (2.12)$$

$$[\sigma\Delta \tau\hat{R}_\tau^\lambda]_\mu$$

$$(\delta T_{\mu\sigma}^\lambda) \begin{bmatrix} 2\delta_\lambda & -T_{\lambda\tau}^0 R^{\tau\mu} \end{bmatrix}$$

with

$$X_{(3)}^\lambda = X_{(3)}^\lambda(g, \delta\Gamma) = 2F_{\sigma\mu}^\lambda \hat{R}^{\lambda\sigma} + (\delta T_{\mu\sigma}^\sigma) \hat{R}^{\lambda\mu} + (\delta T_{\mu\sigma}^\lambda) \hat{R}^{\mu\sigma} \quad (2.13)$$

For families of torsion-free connections they reduce to:

$$\delta(\|\hat{Ric}\|^2) = \nabla_\lambda \check{X}_{(3)}^\lambda - 2h^{\mu\nu} \hat{R}_{\mu\sigma} \hat{R}_\nu^\sigma - 2F_{\mu\sigma}^\lambda \nabla_\rho \hat{R}^{\rho\mu} \quad (2.14)$$

and

$$\check{X}_{(3)}^\lambda = \check{X}_{(3)}^\lambda(g, \delta\Gamma) = 2F_{\mu\sigma}^0 \hat{R}^{\lambda\mu} \quad (2.15)$$

Combining (2.8) together with (2.12) or (2.10) together with (2.14) for torsion-free connections) gives then the variation

$$\delta(\|\hat{Ric}\|^2)$$

Finally, we have:

$$\delta(\|\hat{Riem}\|^2) = \nabla_\lambda X_{(4)}^\lambda - [2F_{\mu\sigma}^\lambda + \delta T_{\mu\sigma}^\lambda] [2\nabla_\lambda R^{\mu\nu\sigma} + T_{\nu\epsilon}^\sigma R_{\lambda}^{\mu\nu\epsilon}] + \quad (2.16)$$

$$h^{\epsilon\gamma} [R_{\epsilon\nu\sigma}^\lambda R_{\lambda\gamma}^{\nu\sigma} + 2R_{\mu\epsilon\sigma}^\lambda R_{\lambda\gamma}^{\mu\sigma} - R_{\epsilon\mu\nu\sigma} R_{\mu\nu\sigma}^\gamma]$$

with

$$X_{(4)}^\lambda = X_{(4)}^\lambda(g, \delta\Gamma) = (2F_{\mu\sigma}^\lambda + \delta T_{\mu\sigma}^\sigma) R_{\nu}^{\mu\lambda\sigma} \quad (2.17)$$

If Γ_ϵ is a family of torsion-free connections (2.16) and (2.17) become, respectively:

$$\delta(\|\hat{Riem}\|^2) = \nabla_\lambda \check{X}_{(4)}^\lambda - 4F_{\mu\sigma}^\lambda \nabla_\nu R_{\lambda}^{\mu\nu\sigma} + \quad (2.18)$$

$$h^{\epsilon\gamma} [R_{\lambda\nu\sigma}^\lambda R_{\lambda\gamma}^{\nu\sigma} + 2R_{\lambda\epsilon\sigma}^\lambda R_{\lambda\gamma}^{\mu\sigma} - R_{\lambda\mu\nu\sigma} R_{\lambda}^{\mu\nu\sigma}]$$

and

$$\check{X}_{(4)}^\lambda = \check{X}_{(4)}^\lambda (g, \delta\Gamma) = 4F_{\mu\sigma}^\mu R_{\nu\dots}^{\mu\lambda\sigma} \quad (2.19)$$

From now on we suppose that Γ_ε is the Levi-Civita connection of g_ε , for each $\varepsilon \in]-a, a[$ so that

$$\Gamma_{\beta\gamma}^\alpha = \check{\Gamma}_{\beta\gamma}^\alpha, T = 0, \nabla_\varepsilon g_\varepsilon = 0, Ric(g) = \check{Ric}(g) = r_{\mu\nu} dx^\mu \otimes dx^\nu$$

and $\hat{Ric}(g) = 0$

Let us set

$$\phi = F_g(\delta g) = \check{\mathcal{F}}[g^*, \nabla(\delta g)] \quad (2.20)$$

Then, from the above identities for the Palatini case we easily obtain the well known formulae for the first variation of the curvature functionals. These are listed hereafter.

For the scalar curvature [see (2.6) and (2.7) respectively] we have:

$$\delta r = \nabla_\lambda \xi_{(1)}^\lambda + h^{\mu\nu} r_{\mu\nu} \quad (2.21)$$

with

$$\xi_{(1)}^\lambda = \check{X}_{(1)}^\lambda (g, \delta\Gamma_g) = g^{\mu\nu} a_{\mu\nu}^\lambda \quad (2.22)$$

having set $r = R(g) = \mathfrak{R}(g, \Gamma_g) = g^{mn} r_{mn}(g)$ and $\alpha = \langle (g, h) = \mathcal{A}[\check{\mathcal{F}}_g(h)]$

Furthermore:

$$\delta(\|Ric(g)\|^2) = \nabla_\lambda \xi_{(2)}^\lambda + h^{\mu\nu} e_{\mu\nu}(\|Ric(g)\|^2) \quad (2.23)$$

with

$$\xi_{(2)}^\lambda = X_{(2)}^\lambda (g, \delta\Gamma_g) + h_{\gamma\sigma} [g^{\mu\lambda} \nabla_\mu r^{\gamma\sigma} + g^{\gamma\sigma} \nabla_\mu r^{\mu\lambda} - 2g^{\mu\lambda} \nabla_\mu r^{\lambda\sigma}] \quad (2.24)$$

where we have set

$$e_{\mu\nu}(\|Ric(g)\|^2) = 2r_{\mu\sigma} r_\mu^\sigma - g^{\alpha\lambda} \nabla_\alpha [2\mathcal{B}(\nabla_\nu r_{\lambda\nu}) - \mathcal{B}(g_{\nu\mu} \partial_\lambda r)] \quad (2.25)$$

Finally, for the energy of the Riemannian curvature $Riem(g) = r^\lambda_{\mu\nu\rho} \partial_\lambda \otimes dx^\mu \otimes dx^\nu \otimes dx^\rho$ we obtain:

$$\delta(\|Riem(g)\|^2) = \nabla_\lambda \xi^\lambda_{(4)} + h^{\mu\nu} e_{\mu\nu} (\|Riem(g)\|^2) \quad (2.26)$$

with

$$\xi^\lambda_{(4)} = X_{(4)}(g, \delta\Gamma_g) - 4h^{\tau\nu} \nabla_\mu r^{\lambda\mu}_{\tau\nu} \quad (2.27)$$

and

$$e_{\mu\nu}(\|Riem(g)\|^2) = 2r_{\mu\alpha\beta\sigma} r^{\alpha\beta\sigma}_{\nu} - 4\nabla_\rho \nabla_\beta r^{\rho\beta}_{\mu\nu} \quad (2.28)$$

Notice that we have not introduced a vectorfield $\xi^\lambda_{(3)}$, since $X^\lambda_{(3)}$ has no counterpart in the metric case (being $Ric(g)$ already symmetric).

3. GENERALIZED HESSIAN MAPPINGS

Let us now calculate the various generalized Hessian mappings (cfr. [5] and [11]) for the variational problems considered in Section 2 under the «Palatini assumption».

In order to simplify the formulae we assume however that each connection of the family (∇_ϵ) is torsion-free. We also set:

$$\overline{A}^\lambda_{\mu\nu} = \overline{A}(\delta\Gamma^\lambda_{\mu\nu}) = \overline{A}(F^\lambda_{\mu\nu}) \quad (3.1)$$

In agreement with the notation introduced in Section 2, vectorfields with a subscript (a) ($a = 1, \dots, 4$) will respectively refer to the corresponding invariants I_a defined by equation (2.3) above. From calculations analogous to those used to derive (2.4), one easily obtains the basic Hessian mapping for $\delta^2\mathcal{H}$ given by:

$$\delta^2\mathcal{H} = \nabla_\lambda Y^\lambda_{(1)} + (\delta^2 g^{\mu\nu}) \check{R}_{\mu\nu} - \mathcal{A}(\delta^2 \Gamma^\lambda_{\mu\nu}) \nabla_\lambda g^{\mu\nu} + Hess_{\mathcal{H}}(h^{mn}, F^\lambda_{\mu\nu}) \quad (3.2)$$

being:

$$Hess_{\mathcal{H}}(h^{\mu\nu}, F^\lambda_{\mu\nu}) = 2h^{\mu\nu} \nabla_\lambda A^\lambda_{\mu\nu} - A^\lambda_{\mu\nu} g^\nu_\rho \overline{\mathcal{A}}^\lambda_{\mu\nu} \quad (3.3)$$

and

$$Y_{(1)}^\lambda \equiv \check{X}_{(1)}^\lambda(g, \delta^2 \Gamma) = g^{\mu\nu} A(\delta^2 \Gamma_{\mu\nu}^\lambda) \quad (3.4)$$

(As we mentioned above, the subscript (1) means here that we are referring to the invariant $I_1 \equiv \mathcal{H}$).

Integrating by parts half of the first term of (3.3) one gets:

$$Hess_{\mathcal{H}}(h^{\mu\nu}, F_{\mu\nu}^\lambda) = H_{\mathcal{H}}(h^{\mu\nu}, F_{\mu\nu}^\lambda) + \nabla_\lambda U_{(1)}^\lambda \quad (3.5)$$

where we set:

$$H_{\mathcal{H}}(h^{\mu\nu}, F_{\mu\nu}^\lambda) = h^{\mu\nu} \nabla_\lambda A_{\mu\nu}^\lambda - A_{\mu\nu}^\lambda [g^{\nu\rho} \overline{\mathcal{A}}_{\rho\lambda}^\mu + \nabla_\lambda h^{\mu\nu}] \quad (3.6)$$

and

$$U_{(1)}^\lambda \equiv \check{X}_{(1)}^\lambda(\delta g, \delta \Gamma) = h^{\mu\nu} A_{\mu\nu}^\lambda \quad (3.7)$$

The same integration by parts performed on the whole first term of (3.3) gives instead:

$$Hess_{\mathcal{H}}(h^{\mu\nu}, F_{\mu\nu}^\lambda) = \overline{H}_{\mathcal{H}}(h^{\mu\nu}, F_{\mu\nu}^\lambda) + 2\nabla_\lambda U_{(1)}^\lambda \quad (3.8)$$

where we set:

$$\overline{H}_R(h^{\mu\nu}, F_{\mu\nu}^\lambda) = -A_{\mu\nu}^\lambda [g^{\nu\rho} \overline{A}_{\rho\lambda}^\mu + 2\nabla_\lambda h_{\mu\nu}] \quad (3.9)$$

Finally, the generalized Hessian mapping H_R obtained by the standard calculation is the following:

$$Hess_{\mathcal{H}}(h^{\mu\nu}, F_{\mu\nu}^\lambda) = \check{\mathcal{H}}_{\mathcal{H}}(h^{\mu\nu}, F_{\mu\nu}^\lambda) - \nabla_\lambda [Y_{(1)}^\lambda + \delta[\nabla_\lambda (\check{X}_{(1)}^\lambda)]] \quad (3.10)$$

where

$$\check{\mathcal{H}}_R(h^{\mu\nu}, F_{\mu\nu}^\lambda) = h^{\mu\nu} \nabla_\lambda A_{\mu\nu}^\lambda - A_{\mu\nu}^\lambda [\nabla_\lambda h^{\mu\nu} + 2g^{\nu\rho} F_{\rho\lambda}^\mu] \quad (3.11)$$

Analogously, one gets for $\|\check{Ric}\|^2$ the following:

$$\delta^2 \left(\left\| \check{Ric} \right\|^2 \right) = \nabla_\lambda Y_{(2)}^\lambda - 2 \mathcal{L}(\delta^2 \Gamma)_{\mu\nu}^\lambda \nabla_\lambda R^{\mu\nu} + 2(\delta^2 g^{\mu\nu}) \check{R}_{\mu\rho} \check{R}_\nu^\rho + Hess_{\left\| \check{Ric} \right\|^2} (h^{\mu\nu}, F_{\mu\nu}^\lambda) \quad (3.12)$$

with

$$Hess_{\left\| \check{Ric} \right\|^2} (h^{\mu\nu}, F_{\mu\nu}^\lambda) = 2(\nabla_\lambda A_{\mu\sigma}^\lambda) g^{\mu\eta} g^{\sigma\tau} (\nabla_\rho A_{\eta\tau}^\rho) - 2A_{\mu\nu}^\rho \check{R}_{\nu\lambda}^\rho \overline{A}_{\lambda\rho}^\mu + 2h^{\mu\eta} \left[4\check{R}_\nu^\sigma \nabla_\lambda A_{\mu\sigma}^\lambda + h^{\sigma\tau} \check{R}_{\mu\sigma}^\tau \check{R}_{\nu\tau}^\lambda \right] \quad (3.13)$$

and

$$Y_{(2)}^\lambda = \check{X}_{(2)}^\lambda(g, \delta^2 \Gamma) = 2\check{R}^{\mu\sigma} A(\delta^2 \Gamma_{\mu\sigma}^\lambda) \quad (3.14)$$

By a simple integration by parts the above generalized Hessian mapping can be written as follows:

$$Hess_{\left\| \check{Ric} \right\|^2} (h^{\mu\nu}, F_{\mu\eta}^\lambda) = H_{\left\| \check{Ric} \right\|^2} (h^{\mu\nu}, F_{\mu\eta}^\lambda) + \nabla_\lambda Z_{(2)}^\lambda \quad (3.15)$$

where

$$H_{\left\| \check{Ric} \right\|^2} (h^{\mu\nu}, F_{\mu\eta}^\lambda) = 2h^{\mu\nu} \left[4\check{R}_\nu^\sigma \nabla_\lambda A_{\mu\sigma}^\lambda + h^{\sigma\tau} \check{R}_{\mu\sigma}^\tau \check{R}_{\nu\tau}^\lambda \right] - 2A_{\mu\sigma}^\rho \left[\overline{A}_{\lambda\rho}^\mu \check{R}^{\lambda\sigma} + \nabla_\rho \left(g^{\mu\nu} g^{\sigma\tau} \nabla_\lambda A_{\nu\tau}^\lambda \right) \right] \quad (3.16)$$

and

$$Z_{(2)}^\lambda = 2A_{\mu\sigma}^\lambda g^{\mu\nu} g^{\sigma\tau} \nabla_\rho A_{\nu\tau}^\rho \quad (3.17)$$

Alternatively, we can write:

$$Hess_{\left\| \check{Ric} \right\|^2} (h^{\mu\nu}, F_{\mu\eta}^\lambda) = \overline{H}_{\left\| \check{Ric} \right\|^2} (h^{\mu\nu}, F_{\mu\eta}^\lambda) + \nabla_\lambda \overline{Z}_{(2)}^\lambda \quad (3.18)$$

being:

$$\begin{aligned} \overline{H}_{\|\check{\text{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\eta}) &= 2h^{\mu\nu}h^{\sigma\tau}\check{R}_{\mu\sigma}\check{R}_{\nu\tau} - 2A^\rho_{\mu\sigma} \left[\overline{A}^\mu_{\lambda\rho}\check{R}^{\lambda\sigma}_{} \right. \\ &\quad \left. + \nabla_\rho(g^{\mu\nu}g^{\sigma\tau}\nabla_\lambda A^\lambda_{}{}^\nu_{} \check{R}^\sigma_{}{}^\nu_{}) \right] \end{aligned} \quad (3.19)$$

and

$$\overline{Z}^\lambda_{(2)} = Z^\lambda_{(2)} + 8h^{\mu\nu} A^\lambda_{\mu\sigma}\check{R}^\sigma_{}{}^\nu_{} \quad (3.20)$$

A further expression of $Hess_{\|\check{\text{Ric}}\|^2}$ is the following:

$$Hess_{\|\check{\text{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) = \overline{H}_{\|\check{\text{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) + \nabla_\lambda \overline{Z}^\lambda_{(2)} \quad (3.21)$$

where

$$\begin{aligned} \overline{H}_{\|\check{\text{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) &= 2h^{\mu\nu}h^{\sigma\tau}\check{R}_{\mu\sigma}\check{R}_{\nu\tau} + 2(\nabla_\rho A^\rho_{\mu\sigma})g^{\mu\nu}g^{\sigma\tau}(\nabla_\lambda A^\lambda_{}{}^\nu_{}) \\ &\quad - 2A^\rho_{\mu\sigma} \left[\check{R}^{\lambda\sigma}_{}{}^\mu_{} \overline{A}^\mu_{\lambda\rho} + 4\nabla_\rho(h^{\mu\nu}\check{R}^\sigma_{}{}^\nu_{}) \right] \end{aligned} \quad (3.22)$$

and

$$\overline{Z}^\lambda_{(2)} = \overline{Z}^\lambda_{(2)} + 2A^\lambda_{\mu\sigma}g^{\mu\nu}g^{\sigma\tau}\nabla_\rho A^\rho_{}{}^\nu_{} \quad (3.23)$$

Finally, the generalized Hessian mapping $\check{\mathfrak{H}}_{\|\check{\text{Ric}}\|^2}$ obtained by standard calculations is given by:

$$Hess_{\|\check{\text{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) = \check{\mathfrak{H}}_{\|\check{\text{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) - \nabla_\lambda Y^\lambda_{(2)} + \delta \left[\nabla_\lambda X^\lambda_{(2)} \right] \quad (3.24)$$

with

$$\begin{aligned} \hat{\mathcal{S}}_{\|\hat{\mathbf{Ric}}\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda) = & 2h^{\mu\nu} \left[h^{\sigma\tau} \hat{R}_{\mu\sigma} \hat{R}_{\nu\tau} + 2\hat{R}_\nu^\sigma \nabla_\lambda A_{\mu\sigma}^\lambda \right] - 2A_{\mu\sigma}^\lambda \left[2\nabla_\lambda \left(h^{\nu\mu} \hat{R}_\nu^\sigma \right) + \right. \\ & \left. + g^{\mu\nu} g^{\sigma\tau} \nabla_\lambda \nabla_\rho A_{\nu\tau}^\rho + 2(\nabla_\lambda g^{\mu\nu}) g^{\sigma\tau} \nabla_\rho A_{\nu\tau}^\rho + 2\hat{R}^{\sigma\rho} F_{\rho\lambda}^\mu \right] \end{aligned} \quad (3.25)$$

Similar calculations can be performed for the skew-symmetric part of the Ricci tensor. In this case one obtains:

$$\delta^2 \left(\|\hat{\mathbf{Ric}}\|^2 \right) = \nabla^\lambda Y_{(3)}^\lambda - 2(\delta^2 g^{\mu\nu}) \hat{R}^{\mu\rho} \hat{R}_\nu^\sigma - 2(\delta^2 \Gamma_{\sigma\lambda}^\lambda) \nabla_\mu \hat{R}^{\mu\sigma} Hess_{\|\hat{\mathbf{Ric}}\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda) \quad (3.26)$$

being:

$$Hess_{\|\hat{\mathbf{Ric}}\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda) = 2h^{\mu\nu} \left[h^{\sigma\tau} \hat{R}_{\mu\sigma} \hat{R}_{\nu\tau} + 4\hat{R}_\nu^\sigma \nabla_{[\sigma} F_{\mu]}^\lambda \right] + 2(\nabla_\mu F_\sigma^\lambda) g^{\mu\nu} g^{\sigma\tau} \nabla_{[\nu} F_{\tau]}^\rho \quad (3.27)$$

and

$$Y_{(3)}^\lambda = \hat{X}_{(3)}^\lambda(g, \delta^2 \Gamma) = 2\hat{R}^{\mu\lambda} \delta^2 \Gamma_{\mu\rho}^\rho \quad (3.28)$$

Here and in the sequel square brackets denote skew-symmetrization. As for the symmetric part, one obtains then:

$$Hess_{\|\hat{\mathbf{Ric}}\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda) = H_{\|\hat{\mathbf{Ric}}\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda) + \nabla_\lambda Z_{(3)}^\lambda \quad (3.29)$$

with

$$H_{\|\hat{\mathbf{Ric}}\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda) = 2h^{\mu\nu} \left[h^{\sigma\tau} \hat{R}_{\mu\sigma} \hat{R}_{\nu\tau} + 4\hat{R}_\nu^\sigma \nabla_{[\sigma} F_{\mu]}^\lambda \right] - 2\nabla_\mu F_{\sigma\lambda}^\lambda (g^{\mu\nu} g^{\sigma\tau} \nabla_{[\nu} F_{\tau]}^\rho) \quad (3.30)$$

and

$$\overline{Z}_{(3)}^\lambda = 2F_{\sigma\mu}^\mu g^{\sigma\tau} g^{\nu\lambda} \nabla_{[\nu} F_{\tau]}^\rho \quad (3.31)$$

Finally, the standard result obtained in the classical way is:

$$Hess_{\|\hat{\text{Ric}}\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda) = \tilde{\mathfrak{H}}_{\|\hat{\text{Ric}}\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda) - \nabla_\lambda Y_{(3)}^\lambda + \delta[\nabla_\lambda \check{X}_{(3)}^\lambda] \quad (3.32)$$

being:

$$\begin{aligned} \tilde{\mathfrak{H}}_{\|\hat{\text{Ric}}\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda) &= 2h^{\mu\nu} \left[2\hat{R}_\nu^\alpha \nabla_{[\sigma} F_{\mu]}^\lambda{}_{\lambda} + \hat{R}_{\mu\sigma}^\alpha h^{\sigma\tau} \hat{R}_{\nu\tau} \right] + \\ &2F_{\sigma\lambda}^\lambda \left(2\nabla_\mu (h^{\nu[\mu]} \hat{R}_\nu^\sigma) + \hat{R}^{\sigma\rho} F_{\rho\mu}^\mu + \right. \\ &\left. \nabla_\mu (g^{\mu\nu} g^{\sigma\tau} \nabla_{[\tau} F_{\nu]\rho}^\rho) \right] \end{aligned} \quad (3.33)$$

Under the Palatini assumption, the generalized Hessian mappings of the norm $\|Riem\|^2$ of the full curvature tensor are quite complicated. For the sake of brevity they will be not included here but relegated to an Appendix.

Let us now revert to the purely metric case. From now on we suppose then $\nabla_\varepsilon g_\varepsilon = 0$ and we reduce the generalized Hessian mappings to this case. For the scalar curvature we have:

$$\delta^2 r = \nabla_\lambda \eta_{(1)}^\lambda + (\delta^2 g^{\mu\nu}) r_{\mu\nu} + \tilde{H}ess_r(h) \quad (3.34)$$

with

$$\tilde{H}ess_r(h) = 4h^{\mu\nu} \nabla_{[\lambda} \phi_{\mu]\nu}^\lambda + 2\phi_{\nu\rho}^\lambda g^{\mu\alpha} g^{\rho[\mu} \nabla_\lambda h^{\nu]\alpha} \quad (3.35)$$

where ϕ is defined by (2.20) and

$$\eta_{(1)}^\lambda = Y_{(1)}^\lambda \mu(g, \delta^2 \Gamma_g) \quad (3.36)$$

An obvious integration by parts gives:

$$\tilde{H}ess_r(h) = \tilde{H}_r(h) + \nabla_\lambda \zeta_{(1)}^\lambda \quad (3.37)$$

being:

$$\tilde{H}_r(h) = 2h^{\mu\nu}\nabla_{[\lambda}\phi_{\mu]\nu}^\lambda + (\nabla_\lambda h^{\mu\alpha})g_{\alpha\mu}g^{\beta\lambda}g_{\nu[\beta}\nabla_{\rho]}h^{\nu\rho} \quad (3.38)$$

and

$$\zeta_{(1)}^\lambda = 2h^{\nu[\mu}\phi_{\mu\lambda}^{\lambda]} \quad (3.39)$$

We also have:

$$\tilde{Hess}_r(h) = \tilde{H}_r^{(0)}(h) + 2\nabla_\lambda \zeta_{(1)}^\lambda \quad (3.40)$$

where we set:

$$\tilde{H}_r^{(0)}(h) = (\nabla_\rho h^{\alpha\nu})g_{\alpha\mu}\nabla_\nu h^{\mu\rho} - (\nabla_\lambda h^{\alpha\nu})g^{\lambda\tau}(\nabla_\tau h^{\mu\rho})g_{\alpha(\nu}g_{\mu)\rho} \quad (3.41)$$

Finally, the generalized Hessian mapping $\tilde{\mathfrak{H}}_r$ obtained by the standard calculations is given by:

$$\tilde{Hess}_r(h) = \tilde{\mathfrak{H}}_r(h) - \nabla_\lambda \eta_{(1)}^\lambda + \delta[\nabla_\lambda \xi_{(1)}^\lambda] \quad (3.42)$$

with

$$\tilde{\mathfrak{H}}_r(h) = 2h^{\mu\nu}\nabla_{[\lambda}\phi_{\mu]\nu}^\lambda \quad (3.43)$$

To make easier the calculation of $\delta^2(\|Ric(g)\|^2)$ we first set:

$$S_{\mu\nu\sigma\tau} = r_{\mu\sigma}r_{\nu\tau} + g_{\mu\sigma}g^{\rho\alpha}\nabla_\alpha[2\mathfrak{B}(\nabla_\tau r_{\rho\nu}) - \mathfrak{B}(g_{\tau\rho}\partial_\nu r)] \quad (3.44)$$

Then, the Hessian mapping of $\delta^2(\|Ric(g)\|^2)$ is given by:

$$\delta^2(\|Ric(g)\|^2) = \nabla_\lambda \eta_{(2)}^\lambda + (\delta^2 g^{\mu\nu})e_{\mu\nu}(\|Ric(g)\|^2) + \tilde{Hess}_{\|Ric(g)\|^2}(h) \quad (3.45)$$

with

$$\tilde{Hess}_{\|Ric(g)\|^2}(h) = 2h^{\mu\nu}[h^{\sigma\tau}S_{\mu\nu\sigma\tau} - 2\mathcal{A}^*(\nabla_\mu r^{\varepsilon\sigma})g_{\nu\rho}\phi_{\varepsilon\sigma}^\rho + 4r_\nu^\sigma\nabla_\lambda\alpha_{\mu\sigma}^\lambda] + \quad (3.46)$$

$$8\phi_{\sigma\lambda}^\rho\phi_{\rho\tau}^{\lambda\sigma} + 2(\nabla_\rho a_{\mu\sigma}^\rho)g^{\mu\sigma}g^{\sigma\tau}\nabla_\lambda a_{\nu\tau}^\lambda$$

and

$$\eta_{(2)}^\lambda \equiv \tilde{X}_{(2)}^\lambda(g, \delta^2 \Gamma_g) + \delta^2 g_{\gamma\sigma} [g^{\mu\lambda} \nabla_\mu r^{\gamma\sigma} + g^{\gamma\sigma} \nabla_\mu r^{\mu\lambda} - 2g^{\mu\lambda} \nabla_\mu r^{\lambda\sigma}] \quad (3.47)$$

The following generalized Hessian mappings can be thence obtained from equation (3.46). First:

$$\tilde{H}_{\|\text{Ric}\|^2}(h) = \tilde{H}_{\|\text{Ric}\|^2}(h) + \nabla_\lambda \zeta_{(2)}^\lambda, \quad (3.48)$$

where $\zeta_{(2)}^\lambda$ is given by calculating (3.17) for $\Gamma = \Gamma g$, i.e.:

$$\zeta_{(2)}^\lambda = 2a_{\mu\sigma}^\lambda g^{\mu\nu} g^{\sigma\tau} \nabla_\rho \alpha_{\nu\tau}^\rho \quad (3.49)$$

$$\tilde{H}_{\|\text{Ric}(g)\|^2}(h) = 2h^{\mu\nu} [h^{\sigma\tau} S_{\mu\nu\sigma\tau} - 2\mathfrak{A}^*(\nabla_\mu r^{\varepsilon\sigma}) g_{\nu\rho} \phi_{\varepsilon\sigma}^\rho +$$

and

$$4r^\sigma \nabla_\lambda a_{\mu\sigma}^\lambda] + 8\phi_{\sigma\lambda}^\rho \phi_{\sigma\lambda}^\lambda [\tau_\tau r^{\lambda}]^\sigma - 2a_{\mu\sigma}^\rho g^{\mu\nu} g^{\sigma\tau} \nabla_\rho \nabla_\lambda a_{\nu\tau}^\lambda \quad (3.50)$$

Moreover we have:

$$\tilde{H}_{\|\text{Ric}(g)\|^2}(h) = \tilde{H}_{\|\text{Ric}(g)\|^2}^{(0)}(h) + \nabla \tilde{\zeta}_{(2)}^\lambda \quad (3.51)$$

With

$$\tilde{H}_{\|\text{Ric}\|^2}(h) = 2h^{\mu\nu} [h^{\sigma\tau} S_{\mu\nu\sigma\tau} - 2\mathfrak{A}^*(\nabla_\mu r^{\varepsilon\sigma}) g_{\nu\rho} \phi_{\varepsilon\sigma}^\rho] - \quad (3.52)$$

$$8a_{\mu\sigma}^\rho \nabla_\rho (h^{\mu\nu} r_\nu^\sigma) + 8\phi_{\sigma\lambda}^\rho \phi_{\rho\tau}^\lambda [\tau_\tau r^{\lambda}]^\sigma +$$

$$2(\nabla_\rho a_{\mu\sigma}^\rho) g^{\mu\nu} g^{\sigma\tau} \nabla_\lambda a_{\nu\tau}^\lambda$$

and

$$\tilde{\zeta}_{(2)}^\lambda = 8a_{\mu 0}^\lambda h^{\mu\nu} r_\nu^\sigma \quad (3.53)$$

Furthermore we find:

$$\tilde{H}_{\|\text{Ric}(g)\|^2}(h) = \tilde{H}_{\|\text{Ric}(g)\|^2}^*(h) + \nabla_\lambda \tilde{\zeta}_{(2)}^\lambda + \nabla_\lambda \tilde{\zeta}_{(2)}^\lambda \quad (3.54)$$

where we set:

$$\begin{aligned} \tilde{H}_{\|Ric(g)\|^2}^{(*)}(h) &= 2h^{\mu\nu}[h^{\sigma\tau}S_{\mu\nu\sigma\tau} - 2\mathcal{A}^*(\nabla_\mu r^{\epsilon\sigma})g_{\nu\rho}\phi_{\epsilon\sigma}^\rho] + \\ &+ 8\phi_{\alpha\lambda}^\rho\phi_{\rho\tau}^{[\tau}r^{\lambda]\sigma} - 8a_{\mu\sigma}^\rho\nabla_\rho(h^{\mu\nu}r_\nu^\sigma) - \\ &2a_{\mu\sigma}^\rho g^{\mu\nu}g^{\sigma\tau}\nabla_\rho\nabla_\lambda a_{\nu\tau}^\lambda \end{aligned} \quad (3.55)$$

Finally, performing the standard calculations one obtains the following:

$$\tilde{H}_{\|Ric(g)\|^2}^{ess}(h) = \tilde{\xi}_{\|Ric(g)\|^2}(h) - \nabla_\lambda \eta_{(2)}^\lambda + \delta(\nabla_\lambda \xi_{(2)}^\lambda) \quad (3.56)$$

where $\tilde{\xi}_{\|Ric(g)\|^2}(h)$ is given by:

$$\begin{aligned} \tilde{\xi}_{\|Ric(g)\|^2}(h) &= 2h^{\gamma\epsilon}\{h^{\sigma\tau}S_{\gamma\epsilon\sigma\tau} - 2[\mathcal{A}^*(\nabla_\gamma r^{\mu\sigma})g_{\epsilon\rho}\phi_{\mu\sigma}^\rho + \\ &a_{\gamma\sigma}^\lambda\nabla_\lambda r_\epsilon^\sigma - r_\epsilon^\sigma\nabla_\lambda a_{\gamma\sigma}^\lambda]\} - 2a_{\mu\sigma}^\lambda k_{\lambda}^{\mu\sigma} \end{aligned} \quad (3.57)$$

being:

$$k_{\rho}^{\mu\sigma} \equiv 2r_\tau^\sigma\nabla_\rho h^{\mu\tau}g^{\mu\epsilon}g^{\sigma\tau}\nabla_\rho\nabla_\gamma a_{\epsilon\tau}^\gamma + 2\phi_{\gamma\rho}^\mu r^{\gamma\sigma} \quad (3.58)$$

In the case of the Levi-Civita connection, the energy $\|Riem(g)\|^2$ is more handy (cfr. Appendix). We have in fact:

$$\delta^2(\|Riem(g)\|^2) = \nabla_\lambda \zeta_{(4)}^\lambda + (\delta^2 g^{\mu\nu})e_{\mu\nu}(\|Riem(g)\|^2) + \tilde{H}_{\|Riem(g)\|^2}^{ess}(h) \quad (3.59)$$

with

$$\begin{aligned} \tilde{H}_{\|Riem(g)\|^2}^{ess}(h) &= 2h^{\epsilon\gamma}\{h^{\sigma\tau}[g_{\tau\epsilon}(r_{\mu\sigma\nu\lambda}r^{\mu\nu\lambda}_{\gamma\dots} + \\ &4g^{\mu\rho}g^{\nu\beta}\nabla_\mu\nabla_\nu r_{\beta\gamma\rho}) + 4r_{\mu\epsilon\alpha\sigma}g^{\alpha\lambda}g^{\mu\rho}r_{\tau\rho\gamma\lambda}] + \\ &4[\nabla_\nu\phi_{\sigma\nu}^\alpha g_{\alpha\epsilon}g^{\mu\rho} - 2\phi_{\epsilon\sigma}^\rho]r_{\gamma\rho\dots}^{\nu\sigma}\} - \\ &8\phi_{\epsilon\nu}^\gamma\phi_{\sigma\mu}^\alpha g_{\gamma\alpha}r^{\epsilon\mu\nu\sigma}_{\dots} + 8(\nabla_\nu\phi_{\mu\sigma}^\lambda)g_{\lambda\alpha}g^{\mu\rho}g^{\sigma\tau}g^{\nu\beta}\nabla_{[\beta}\phi_{\rho]}^\alpha \end{aligned} \quad (3.60)$$

and

$$\zeta_{(4)}^\lambda = 4[\nabla_\mu(\delta h_{\varepsilon\sigma})]r^{\varepsilon\mu\lambda\sigma}_{\dots} - 4(\delta h_{\sigma\varepsilon})\nabla_\mu r^{\varepsilon\mu\lambda\sigma}_{\dots} \quad (3.61)$$

Finally, we have:

$$\tilde{Hess}_{\|Riem(g)\|^2}(h) + \tilde{H}^{(0)}_{\|Riem(g)\|^2}(h) + \nabla_\lambda \tilde{\zeta}_{(4)}^\lambda \quad (3.62)$$

being:

$$\tilde{H}^{(0)}_{\|Riem(g)\|^2}(h) = -8\phi_{\varepsilon\nu}^\gamma \tilde{k}_\gamma^{\varepsilon\nu} + 2h^{\varepsilon\gamma} \{h^{\sigma\tau} [g_{\tau\varepsilon} (r_{\mu\sigma\nu\lambda} r_{\gamma\dots}^{\mu\nu\gamma} + \quad (3.63)$$

$$4g^{\mu\rho} g^{\nu\beta} \nabla_\mu \nabla_\nu r_{\beta\gamma\sigma\rho}) + 4r_{\mu\varepsilon\alpha\sigma} g^{\alpha\gamma} g^{\mu\rho} r_{\tau\rho\gamma\lambda}] +$$

$$4[\nabla_\nu (\phi_{\sigma\mu}^\alpha g_{\alpha\varepsilon} g^{\mu\rho} - 2\phi_{\varepsilon\sigma}^\rho) r_{\gamma\rho\dots}^{\nu\sigma}],$$

where:

$$k_\gamma^{\varepsilon\nu} = \phi_{\sigma\mu}^\alpha g_{\gamma\alpha} r^{\varepsilon\mu\nu\sigma}_{\dots} + g_{\gamma\alpha} g^{\varepsilon\rho} g^{\nu\tau} g^{\sigma\beta} \nabla_\sigma \nabla_{[\beta} \phi_{\tau]}^\alpha{}_\rho \quad (3.64)$$

and

$$\zeta_{(4)}^\lambda = 8\phi_{\mu\sigma}^\nu g_{\nu\alpha} g^{\mu\rho} g^{\sigma\tau} g^{\lambda\beta} \nabla_{[\beta} \phi_{\tau]}^\alpha{}_\rho \quad (3.65)$$

4. GENERALIZED JACOBI MAPPINGS

From the generalized Hessian mappings deduced in Section 3, we can easily obtain the generalized Jacobi mappings and the generalized Jacobi equations, following the discussion of [5]. Recall that a generalized Jacobi equation is obtained from a generalized Jacobi mapping by requiring that the tensorfields of type (0,2) and (1,2) which appear as the coefficient of h and F in the generalized Jacobi mapping vanish «on shell», and considering the ensuing conditions as partial differential equations in the variables $h^{\mu\nu}$ and $F^{\lambda}_{\mu\nu}$. As a consequence, let us remark that, in particular, all the generalized Hessian mappings which are already linear in the arguments and can be directly considered as generalized Jacobi mappings.

We start by considering first the Palatini assumption. Thence, according to our previous remark, equations (3.3), (3.6), (3.9) and (3.11) are already the Jacobi mappings for the scalar curvature \mathfrak{R} . From the generalized Hessian mapping (3.13) we obtain the following generalized Jacobi mapping for $\|\tilde{\check{Ric}}\|^2$:

$$Hess_{\|\tilde{\check{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) = J^{(1)}_{\|\tilde{\check{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) + \nabla_\lambda Z^\lambda_{(2)} \quad (4.1)$$

where $Z^\lambda_{(2)}$ is given by (3.17) and $J^{(1)}_{\|\tilde{\check{Ric}}\|^2}$

is given by:

$$\begin{aligned} J^{(1)}_{\|\tilde{\check{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) &= 2h^{\mu\nu}[4\check{R}^\sigma_\nu \nabla_\lambda A^\lambda_{\mu\sigma} + h^{\sigma\tau}\check{R}_{\mu\sigma}\check{R}^\sigma_{\nu\tau}] - \\ &2A^\rho_{\mu\sigma}[\overline{A}^\mu_{\lambda\rho}\check{R}^{\lambda\sigma} + \nabla_\rho(g_{\mu\nu}g^{\sigma\tau}\nabla_\lambda A^\lambda_{\nu\tau})] = \\ &H_{\|\tilde{\check{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) \end{aligned} \quad (4.2)$$

Also equations (3.19) and (3.25) are generalized Jacobi mappings. Finally, from equation (3.22) it follows

$$\stackrel{=}{H}_{\|\tilde{\check{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) = J^{(2)}_{\|\tilde{\check{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) + \nabla_\lambda Z^\lambda_{(2)} \quad (4.3)$$

with:

$$\begin{aligned} J^{(2)}_{\|\tilde{\check{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) &= 2h^{\mu\nu}h^{\sigma\tau}\check{R}_{\mu\sigma}\check{R}^\sigma_{\nu\tau} - 2A^\rho_{\mu\sigma}[\check{R}^{\lambda\sigma}\overline{A}^\mu_{\lambda\rho} + \\ &4\nabla_\rho(h^{\mu\nu}\check{R}^{\sigma\nu}) + \nabla_\rho(g^{\mu\nu}g^{\sigma\tau}\nabla_\lambda A^\lambda_{\nu\tau})] \end{aligned} \quad (4.4)$$

For the skew-symmetric part $\|\hat{\check{Ric}}\|^2$ we have instead:

$$Hess_{\|\hat{\check{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) = J^{(1)}_{\|\hat{\check{Ric}}\|^2}(h^{\mu\nu}, F^\lambda_{\mu\nu}) + \nabla_\lambda Z^\lambda_{(3)} \quad (4.5)$$

where $Z_{(3)}^\lambda$ is given by (3.31) and we set

$$J_{\|Ric\|^2}^{(1)}(h^{\mu\nu}, F_{\mu\nu}^\lambda) = H_{\|Ric\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda) \quad (4.6)$$

which together with $\overline{H}_{\|Ric\|^2}(h^{\mu\nu}, F_{\mu\nu}^\lambda)$ are generalized Jacobi mappings. Under the Platini assumption handy generalized Jacobi equations cannot be obtained; their complicated expression will therefore be given in the Appendix.

Let us now revert to the purely metric case and assume that Γ_ε is the Levi-Civita connection of g_ε for each $\varepsilon \in]-a, a[$. Then, for the scalar curvature r we have:

$$\tilde{Hess}_r(h) = [\tilde{J}(r)]_{\mu\nu} h^{\mu\nu} + \nabla_\lambda W_{(1)}^\lambda \quad (4.7)$$

with

$$[\tilde{J}(r)]_{\mu\nu} \equiv [\tilde{J}(r)]_{\mu\nu} (\nabla^2 h) \equiv 2g^{\alpha\rho} \nabla_\lambda (g_{\mu(\nu} \phi_{\alpha\rho)}^\lambda) - 2\nabla_\mu \phi_{\lambda\nu}^\lambda \quad (4.8)$$

or, equivalently:

$$[\tilde{J}(r)]_{\mu\nu} = g^{\lambda\rho} (\nabla_\lambda \nabla_\rho h^{\varepsilon\gamma}) g_{\nu(\mu} g_{\varepsilon)\gamma} - 2\nabla_\lambda \nabla_{(\mu} g_{\gamma)\nu} h^{\gamma\lambda} + g_{\gamma\lambda} \nabla_\mu \nabla_\nu h^{\lambda\gamma} \quad (4.9)$$

here we have set:

$$W_{(1)}^\lambda = 2h_{\alpha\mu} g^{\alpha[\mu} g^{\nu]\rho} \phi_{\nu\rho}^\lambda \quad (4.10)$$

We have also:

$$\tilde{Hess}_r(h) = [\tilde{J}^{(1)}(r)]_{\mu\nu} h^{\mu\nu} + \nabla_\lambda W_{(2)}^\lambda \quad (4.11)$$

being:

$$[\tilde{J}^{(1)}(r)]_{\mu\nu} \equiv [\tilde{J}^{(1)}(r)]_{\mu\nu} (\nabla^2 h) = g_{\mu\nu} g^{\beta\lambda} \nabla_\lambda (g_{\alpha[\rho} \nabla_{\beta]} h^{\alpha\rho}) - 2\nabla_{[\mu} \phi_{\lambda]\nu}^\lambda \quad (4.12)$$

and

$$W_{(2)}^\lambda = \zeta_{(1)}^\lambda + h_{\alpha\mu} g^{\alpha\mu} g^{\beta\lambda} g^{\nu\rho} \nabla_{[\rho} h_{\beta]\nu} \quad (4.13)$$

A further generalized Jacobi mapping is obtained by setting:

$$\tilde{Hess}_R(h) = [\tilde{J}^{(2)}(r)]_{\mu\nu} h^{\mu\nu} + \nabla_\lambda W_{(3)}^\lambda, \quad (4.14)$$

with

$$[\tilde{J}^{(2)}(r)]_{\mu\nu} \equiv [\tilde{J}^{(2)}(r)]_{\mu\nu} (\nabla^2 h) = \nabla_\lambda [g^{\lambda\tau} (\nabla_\tau h^{\alpha\rho}) g_{\mu(\nu} g_{\alpha\rho)} - g_{\rho\mu} \nabla_\nu h_{\lambda\rho}] \quad (4.15)$$

and

$$W_{(3)}^\lambda = 2\mathcal{E}_{(1)}^\lambda + W_{(4)}^\lambda \quad (4.16)$$

being:

$$W_{(4)}^\lambda = h^{\alpha\nu} [g_{\alpha\mu} \nabla_\nu h^{\mu\lambda} - g^{\lambda\tau} (\nabla_\tau h^{\mu\rho}) g_{\alpha(\nu} g_{\mu\rho)}] \quad (4.17)$$

Finally, the usual calculations give:

$$\tilde{Hess}_r(h) = [\tilde{J}^{(3)}(r)]_{\mu\nu} h^{\mu\nu} + \delta \nabla_\lambda \eta_{(1)}^\lambda \quad (4.18)$$

being:

$$[\tilde{J}^{(3)}(r)]_{\mu\nu} \equiv [\tilde{J}^{(3)}(r)]_{\mu\nu} (\nabla^2 h) = \nabla_\lambda \alpha_{\mu\nu}^\lambda = 2\nabla_{[\lambda} \nabla_{\phi\nu]}^\lambda \quad (4.19)$$

For the Lagrangian $\|Ric(g)\|^2$

we find:

$$\tilde{Hess}_{\|Ric(g)\|^2}(h) = [\tilde{J}_{\|Ric(g)\|^2}]_{\mu\nu} h^{\mu\nu} + \nabla_\lambda W_{(5)}^\lambda \quad (4.21)$$

where

$$[\tilde{J}_{\|Ric(g)\|^2}]_{\mu\nu} \equiv [\tilde{J}_{\|Ric(g)\|^2}]_{\mu\nu} (\nabla^2 h) = \quad (4.21)$$

$$\begin{aligned} &= 2h^{\sigma\tau} S_{\mu\nu\sigma\tau} - 4g_{\nu\tau} \phi_{\varepsilon\sigma}^\tau \mathcal{A}^*(\nabla_\mu r^{\sigma\varepsilon}) + 8r_\nu^\sigma \nabla_\lambda \alpha_{\mu\sigma}^\lambda + \\ &4g_{\mu\nu} \nabla_\lambda (\phi_{\sigma\tau}^\lambda r^{\sigma\tau}) + 4g_{\mu\beta} g^{\nu\lambda} \nabla_{[\lambda} (r_{\nu]} \phi_{\sigma\tau}^\beta) - \end{aligned}$$

$$4g_{\tau\nu}\nabla_\lambda(\phi_{\sigma\mu\tau}^{(\lambda}r^{\tau)}_{\sigma}) + g^{\alpha\gamma}[g_{\mu\nu}g^{\varepsilon\tau}\nabla_\gamma\nabla_\varepsilon\nabla_\rho\alpha_{\tau\alpha}^\rho + \\ \nabla_\gamma\nabla_\alpha\nabla_\rho\alpha_{\mu\nu}^\rho - 2\nabla_\gamma\nabla_\mu\nabla_\rho\alpha_{\alpha\nu}^\rho]$$

and

$$W_{(5)}^\lambda = \zeta_{(2)}^\lambda + W_{(6)}^\lambda + W_{(7)}^\lambda \quad (4.22)$$

having set:

$$W_{(6)}^\lambda = 4h_{\mu\nu}\mathfrak{B}^*(g^{\lambda\beta}g^{\mu[\nu}g^{\tau]\rho}\nabla^\tau\nabla_\gamma\alpha_{\beta\rho}^\gamma) \quad (4.23)$$

and

$$W_{(7)}^\lambda = 8h_{\mu\nu}\mathfrak{B}^*(\phi_{\sigma\tau}^\lambda g^{\mu\mu[\nu}r^{\tau]\sigma}) \quad (4.24)$$

Equation (4.21) is equivalent to:

$$[\tilde{J}_{\|Ric(g)\|^2}]_{\mu\nu} = 2h^{\sigma\tau}S^{\mu\nu\sigma\tau} - 4g_{\nu\tau}\phi_{\varepsilon\sigma}^\tau\mathfrak{A}^*(\nabla_\mu r^{\sigma\varepsilon}) + 8r_\nu^\sigma\nabla_\lambda\alpha_{\mu\sigma}^\lambda + \\ 4g_{\mu\nu}g_{\nu\alpha}\nabla_\lambda\{\mathfrak{B}^*(g^{\lambda\beta}g^{\varepsilon[\alpha}g^{\tau]\rho}\nabla_\tau\nabla_\gamma\alpha_{\beta\rho}^\gamma + 2\phi_{\sigma\tau}^\lambda g^{\varepsilon[\alpha}r^{\tau]\sigma})\}$$

Moreover we have:

$$\tilde{Hess}_{\|Ric(g)\|^2}(h) = [\tilde{J}_{\|Ric(g)\|^2}^{(1)}]_{\mu\nu} + \nabla_\lambda W_{(8)}^\lambda \quad (4.26)$$

being:

$$[\tilde{J}_{\|Ric(g)\|^2}^{(1)}]_{\mu\nu} \equiv [\tilde{J}_{\|Ric(g)\|^2}^{(1)}]_{\mu\nu}(\nabla^2 h) = \\ 2h^{\sigma\tau}S_{\mu\nu\sigma\tau} - 4g_{\nu\tau}\phi_{\varepsilon\sigma}^\tau\mathfrak{A}^*(\nabla_\mu r^{\sigma\varepsilon}) + 8r_\nu^\sigma\nabla_\lambda\alpha_{\mu\sigma}^\lambda + \\ 4g_{\mu\varepsilon}g_{\nu\alpha}\nabla_\lambda\{\mathfrak{B}^*(g^{\lambda\beta}g^{\varepsilon[\alpha}g^{\tau]\rho}\nabla_\tau\nabla_\gamma\alpha_{\beta\rho}^\gamma + 2\phi_{\sigma\tau}^\lambda\phi_{\rho\tau}^{\alpha\lambda}g^{\rho\varepsilon})\}$$

And

$$W_{(8)}^\lambda = \zeta_{(2)}^\lambda + W_{(6)}^\lambda + W_{(9)}^\lambda \quad (4.28)$$

having set:

$$W_{(9)}^\lambda = 8h_{\mu\nu}\mathfrak{B}^*(r^{\nu[\lambda}\phi_{\rho\tau}^{\lambda]}\alpha^{\rho\mu}) \quad (4.29)$$

Furthermore the following holds:

$$\tilde{Hess}_{\|\text{Ric}(g)\|^2}(h) = [\tilde{J}_{\|\text{Ric}(g)\|^2}^{(2)}]_{\mu\nu} h^{\mu\nu} + \nabla_\lambda W_{(10)}^\lambda \quad (4.30)$$

being:

$$[\tilde{J}_{\|\text{Ric}(g)\|^2}^{(2)}]_{\mu\nu} = [\tilde{J}_{\|\text{Ric}(g)\|^2}^{(2)}]_{\mu\nu}(\nabla^2 h) = \quad (4.31)$$

$$\begin{aligned} & 2h^{\sigma\tau} S_{\mu\nu\sigma\tau} - 4g_{\nu\tau} \phi_{\varepsilon\sigma}^\tau \mathcal{A}^*(\nabla_\mu r^{\sigma\varepsilon}) + 4g_{\mu\nu} \nabla_\lambda \nabla_\sigma (h^{\tau(\sigma} r^{\lambda)}) + \\ & 4g_{\mu\varepsilon} g_{\nu\alpha} \nabla_\lambda \{ \mathfrak{B}^*(g^{\lambda\beta} g^{\varepsilon[\alpha} g^{\tau]\rho} \nabla_\tau \nabla_\gamma \alpha_{\beta\rho}^\gamma + \\ & 2\phi_{\sigma\tau}^\lambda g^{\varepsilon[\alpha} r^{\tau]\sigma} - 2g^{\varepsilon\rho} \nabla_\rho (r^\alpha_\sigma h^{\sigma\lambda}) \} \end{aligned}$$

and

$$W_{(10)}^\lambda = \zeta_{(2)}^\lambda + \zeta_{(2)}^\lambda + W_{(6)}^\lambda + W_{(7)}^\lambda + W_{(11)}^\lambda \quad (4.32)$$

having set:

$$W_{(11)}^\lambda = 4h_{\mu\nu} [g^{\mu\nu} \nabla_\sigma (h^{\tau(\sigma} r^{\lambda)}) - 2\mathfrak{B}^*(g^{\rho\mu} \nabla_\rho (r^\nu_\sigma h^{\sigma\lambda}))] \quad (4.33)$$

By two obvious integrations by parts two further generalized Jacobi mappings can be obtained, namely:

$$\tilde{Hess}_{\|\text{Ric}(g)\|^2}(h) = [\tilde{J}_{\|\text{Ric}(g)\|^2}^{(3)}]_{\mu\nu} h^{\mu\nu} + \nabla_\lambda W_{(12)}^\lambda \quad (4.34a)$$

$$\tilde{Hess}_{\|\text{Ric}(g)\|^2}(h) = [\tilde{J}_{\|\text{Ric}(g)\|^2}^{(4)}]_{\mu\nu} h^{\mu\nu} + \nabla_\lambda W_{(13)}^\lambda \quad (4.34b)$$

The quantities appearing in (4.34a) and (4.34b) are the following:

$$[\tilde{J}_{\|\text{Ric}(g)\|^2}^{(3)}]_{\mu\nu} = [\tilde{J}_{\|\text{Ric}(g)\|^2}^{(3)}]_{\mu\nu}(\nabla^2 h) = \quad (4.35)$$

$$\begin{aligned} & 2h^{\sigma\tau} S_{\mu\nu\sigma\tau} - 4g_{\nu\tau} \phi_{\varepsilon\sigma}^\tau \mathcal{A}^*(\nabla_\mu r^{\sigma\varepsilon}) + \\ & 4g_{\mu\nu} \nabla_\lambda \nabla_\sigma (h^{\tau(\sigma} r^{\lambda)}) + \\ & 4g_{\mu\varepsilon} g_{\nu\alpha} \nabla_\lambda \{ \mathfrak{B}^*(g^{\lambda\beta} g^{\varepsilon[\alpha} g^{\tau]\rho} \nabla_\tau \nabla_\gamma \alpha_{\beta\rho}^\gamma - \\ & 2g^{\varepsilon\rho} \nabla_\rho (r^\alpha_\sigma h^{\sigma\lambda}) + 2r^\alpha_{[\lambda} \phi_{\rho\tau}^{\tau]} g^{\rho\varepsilon} \} \end{aligned}$$

$$[\tilde{J}_{\|Ric(g)\|^2}^{(4)}]_{\mu\nu} = [\tilde{J}_{\|Ric(g)\|^2}^{(4)}]_{\mu\nu}(\nabla^2 h) = \quad (4.36)$$

$$\begin{aligned} & 2h^{\sigma\tau}S_{\mu\nu\sigma\tau} + 8r_v^\sigma\phi\nabla_\lambda\alpha_{\mu\sigma}^\lambda + \\ & 4g_{\mu\epsilon}g_{\nu\alpha}\nabla_\lambda\{\mathfrak{B}^*(g^{\lambda\beta}g^{\epsilon[\alpha}g^{\tau]\rho}\nabla_\tau\nabla_\gamma\alpha_{\beta\rho}^\gamma + \\ & 2h^{\epsilon[\alpha}\nabla_\sigma r^{\sigma]\lambda} + 2\phi_{\sigma\tau}^\lambda g^{\epsilon[\alpha}r^{\tau]\sigma})\} \end{aligned}$$

$$W_{(12)}^\lambda = \zeta_{(2)}^\lambda + W_{(8)}^\lambda + W_{(11)}^\lambda \quad (4.37)$$

and

$$W_{(12)}^\lambda = W_{(5)}^\lambda + W_{(14)}^\lambda \quad (4.38)$$

being:

$$W_{(14)}^\lambda = 8h_{\mu\nu}\mathfrak{B}^*(h^{\mu[\nu}\nabla_\sigma r^{\sigma]\lambda}) \quad (4.39)$$

Finally, standard calculations

$$\tilde{Hess}_{\|Ric(g)\|^2}(h) = [\tilde{J}_{\|Ric(g)\|^2}^{(5)}]_{\mu\nu}h^{\mu\nu} + \delta\nabla_\lambda\xi_{(2)}^\lambda - \nabla_\lambda\eta_{(2)}^\lambda + \nabla_\lambda W_{(15)}^\lambda, \quad (4.40)$$

being:

$$\begin{aligned} & [\tilde{J}_{\|Ric(g)\|^2}^{(5)}]_{\mu\nu} = [\tilde{J}_{\|Ric(g)\|^2}^{(5)}]_{\mu\nu}(\nabla^2 h) = \\ & = 2h^{\sigma\tau}S_{\mu\nu\sigma\tau} - 4[\mathcal{A}^*(\nabla_\mu r^{\mu\sigma})g_{\nu\rho}\phi_{\sigma\gamma}^\rho + \\ & \alpha_{\mu\sigma}^\lambda\nabla_\lambda r_v^\sigma - r_v^\sigma\nabla_\lambda\alpha_{\mu\sigma}^\lambda] - \\ & 4g_{\mu\alpha}g_{\nu\beta}\nabla_\lambda[\mathfrak{B}^*(g^{\alpha\rho}\mathcal{A}^*(k_p^{\lambda\nu}))] \end{aligned} \quad (4.41)$$

with

$$W_{(15)}^\lambda = -4h_{\mu\nu}\mathfrak{B}^*(g^{\mu\rho}A^*(k_p^{\lambda\nu})) \quad (4.42)$$

The simplest generalized Jacobi mapping for $(\|Riem\|^2)$ which can be found is given by:

$$\tilde{Hess}_{\|Riem(g)\|^2}(h) = \nabla_\lambda W_{(16)}^\lambda + \tilde{J}_{\|Riem(g)\|^2}h^{\mu\nu}, \quad (4.43)$$

where:

$$\begin{aligned}
 [\tilde{J}_{\|Riem(g)\|^2}]_{\mu\nu} &= [\tilde{J}_{\|Riem(g)\|^2}]_{\mu\nu}(\nabla^2 h) = \\
 &= 2h^{\varepsilon\gamma}\{h^{\sigma\tau}[g_{\tau\varepsilon}(r_{\mu\sigma\nu\lambda}r_{\gamma\ldots\lambda}^{\mu\nu\lambda} + \\
 &4g^{\mu\rho}g^{\nu\beta}\nabla_\mu\nabla_\nu r_{\beta\gamma\sigma\rho}) + 4r_{\mu\varepsilon\alpha\sigma}g^{\alpha\lambda}g^{\mu\rho}r_{\tau\rho\gamma\lambda}] + \\
 &4[\nabla_\nu\phi_{\sigma\mu}^\alpha g_{\alpha\varepsilon}g^{\mu\rho} - 2\phi_{\varepsilon\sigma}^\rho]r_{\gamma\rho}^{\nu\sigma}\} - \\
 &8g_{\mu\alpha}g_{\nu\beta}\nabla_\lambda[\mathfrak{Z}^*(g^{\alpha\gamma}K_\gamma^{\beta\lambda})]
 \end{aligned} \tag{4.44}$$

and

$$W_{(16)}^\lambda = \zeta_{(4)}^\lambda - 8h_{\mu\nu}\mathfrak{Z}^*(g^{\gamma\mu}K_\gamma^{\nu\lambda}) \tag{4.45}$$

Other forms of the Jacobi mappings can be calculated by different integration by parts as above; however, their expression is rather cumbersome and are not worthy to be reported here.

APPENDIX

As previously announced, in this Appendix we give a Hessian and a Jacobi mapping for $\|Riem\|^2$ in the general case. First, we put:

$$e(\|Riem\|^2)_{\lambda}^{\mu\sigma} = 4\nabla_\nu R_{\lambda\ldots}^{\mu\nu\sigma}, \tag{A. 1}$$

$$e(\|Riem\|^2)_{\varepsilon\gamma} = R_{\varepsilon\nu\sigma}^\lambda R_{\lambda\gamma\ldots}^{\nu\sigma} + 2R_{\mu\varepsilon\sigma}^\lambda R_{\lambda\gamma\ldots}^{\mu\sigma} - R_{\varepsilon\mu\sigma}^\lambda R_{\gamma\ldots}^{\mu\nu\sigma}, \tag{A. 2}$$

$$^{(1)}H_{\lambda\alpha}^{\gamma\mu\sigma\beta\rho\tau} = 8g_{\gamma\alpha}g^{\mu\rho}g^{\nu[\beta}g^{\tau]\sigma}, \tag{A. 3}$$

$$^{(1)}H_{\lambda\gamma\varepsilon}^{\nu\mu\sigma} = 8(R_{\varepsilon\ldots}^{\mu\sigma\nu}g_{\gamma\lambda} - \delta_\gamma^\mu R_{\lambda\ldots}^{\mu\sigma} - 2\delta_\gamma^\mu R_{\gamma(\varepsilon\tau)}^\nu g^{\sigma\tau}) \tag{A. 4}$$

and

$$^{(3)}Heg_{\varepsilon\gamma\mu\rho} = 2(2R_{\mu\varepsilon\sigma}^\lambda 2R_{\lambda\rho\gamma}^\sigma + R_{\sigma\varepsilon\mu}^\lambda R_{\lambda\gamma\rho}^\sigma + \tag{A. 5}$$

$$2R_{\mu\sigma\beta\varepsilon}R_{\rho\gamma\ldots}^{\beta\sigma} - R_{\mu\varepsilon\beta\sigma}R_{\rho\gamma\ldots}^{\beta\sigma} + R_{\mu\alpha\sigma\beta}R_{\gamma\ldots}^{\alpha\sigma\beta}g_{\rho\varepsilon\ldots}).$$

Then the second variation of $\|Riem\|^2$ is given by:

$$\begin{aligned} \delta^2(\|Riem\|^2) &= \nabla_\lambda Y_{(4)}^\lambda + (\delta^2 \Gamma_{\mu\sigma}^\lambda) e(\|Riem\|^2) x^\sigma + \\ &(\delta^2 g^{\varepsilon\gamma}) e(\|Riem\|^2)_{\varepsilon\gamma} + Hess_{\|Riem\|^2} \end{aligned} \quad (A.6)$$

being:

$$\begin{aligned} Hess_{\|Riem\|^2} &= {}^{(1)}H_{\lambda\alpha}^{\nu\mu\sigma\beta\rho\tau} (\nabla_\nu F_{\mu\sigma}^\lambda) (\nabla_\beta F_{\rho\tau}^\alpha) + {}^{(2)}H_{\lambda\gamma\varepsilon}^{\nu\mu\sigma} (\nabla_\nu F_{\mu\sigma}^\lambda) h^{\gamma\varepsilon} + \\ &8R_{\gamma}^{\mu\nu\sigma} F_{\lambda\nu}^\gamma + {}^{(3)}H_{\varepsilon\gamma\mu\rho} h^{\varepsilon\gamma} h^{\mu\rho} \end{aligned} \quad (A.7)$$

and

$$\check{X}_{(4)}^\lambda \equiv Y_{(4)}^\lambda(g, \delta^2 \Gamma) = 4(\delta^2 \Gamma_{\mu\sigma}^\nu) R_{\nu\dots}^{\mu\lambda\sigma} \quad (A.8)$$

Finally, a Jacobi mapping is obtained by putting:

$$J_{\|Riem\|^2} = Hess_{\|Riem\|^2} + \nabla_\nu Z_{(4)}^\lambda \quad (A.9)$$

being:

$$\begin{aligned} J_{\|Riem\|^2} &= h^{\varepsilon\gamma} [{}^{(3)}H_{\varepsilon\gamma\mu\rho} h^{\mu\rho} + {}^{(2)}H_{\lambda\gamma\varepsilon}^{\nu\mu\sigma} \nabla_\nu F_{\mu\sigma}^\lambda] + \\ &F_{\mu\sigma}^\lambda [8R_{\gamma}^{\mu\nu\sigma} F_{\lambda\nu}^\gamma - \nabla_\nu ({}^{(1)}H_{\lambda\alpha}^{\nu\mu\sigma\beta\rho\tau} \nabla_\beta F_{\rho\tau}^\sigma)] \end{aligned} \quad (A.10)$$

and

$$Z_{(4)}^\lambda = {}^{(1)}H_{\nu\alpha}^{\lambda\mu\sigma\beta\rho\tau} F_{\mu\sigma}^\nu \nabla_\beta F_{\rho\tau}^\alpha \quad (A.11)$$

5. CONCLUSIONS

We have been able to calculate, both in the metric-affine («Palatini») formalism and in the purely metric formalism, various expressions for the Hessian and generalized Jacobi equations for Lagrangians quadratic in the curvature. In a forthcoming shorter note [7] we shall apply these formulae to arbi-

trary functions of the scalar curvature and of the above squared norms. All these results should be helpful in calculations concerning the stability and the energy of higher order gravity theories. We hope to address this problem in a further investigation.

SUMMARY

We consider the second variation and the appropriate Jacobi equations for the scalar curvature and the quadratic curvature invariants based on an independent pair (g, T) formed by a metric and torsionless linear connection (so-called «Palatini formalism»). The purely metric case is obtained as a consequence. The results are worked out in full detail in view of applications to non-linear Lagrangian theories of gravitation.

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